DYNAMIC COMPACTNESS OF COUPLED THERMOELASTIC WAVES IN CONTINUOUSLY NONUNIFORM ANISOTROPIC MEDIA

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In the present work, the dynamic problem of coupled thermoelasticity with the most general type of nonuniformity and anisotropy is analyzed. The hyperbolic nature of the system of equations of coupled thermoelasticity is demonstrated, effects of extinction of separate waves by superposition of elastic and thermoelastic wave fronts are investigated, and the interrelationship of different orders of discontinuity of stresses, displacements, and temperature is determined. The case of the uncoupled problem of thermoelasticity is especially analyzed. Sufficient conditions are obtained for the dynamic density for wave processes in thermoelasticity, previously investigated for boundary value problems of hyperbolic systems of second order differential equations [1], and elastic stress waves [2] are obtained. The generally accepted system of tensor notation for the theory of thermoelasticity is used [3].

1. The complete system of equations of linear thermoelasticity, taking into account the interrelationship of temperature and mechanical fields with a finite rate of propagation of the body (see [4]), consists of the following:

Equation of heat balance

$$\nabla_j q^j + c_\varepsilon \dot{\Theta} + \eta T_0 \beta^{ij} \dot{\varepsilon}_{ij} = 0; \tag{1.1}$$

Law for heat conduction

$$\tau \dot{q}^j + q^j = -k^{ij} \Theta_{,i}; \tag{1.2}$$

Equations of motion

$$\rho^{-1} \nabla_j \sigma^{ij} = \ddot{u}^i; \tag{1.3}$$

Cauchy's equations

$$\varepsilon_{ij} = 0.5(\nabla_i u_j + \nabla_j u_i); \tag{1.4}$$

and the Duhamel–Neumann law

$$\sigma^{ij} = C^{ijkl} \nabla_k u_l - \beta^{ij} \Theta \quad \text{or} \quad \varepsilon_{ij} = E_{ijkl} \sigma^{kl} + \alpha_{ij} \Theta. \tag{1.5}$$

For $\tau = 0$ and $\eta = 1$, this system of equations degenerates into the usual system of equations for coupled thermoelasticity with Fourier's law of heat conduction [3], while for $\eta = 0$ we have the uncoupled problem of thermoelasticity. Eliminating from (1.1)-(1.5) the appropriate variables, we obtain the system to be solved for the stresses and temperature:

$$\begin{gathered} -\nabla_j (k^{ij} \nabla_i \Theta) + (c_\varepsilon + \eta T_0 H_1)(\dot{\Theta} + \tau \ddot{\Theta}) + \eta T_0 \alpha_{kl} (\dot{\sigma}^{kl} + \tau \ddot{\sigma}^{kl}) = 0, \\ C^{ljki} \nabla_k \left(\rho^{-1} \nabla_s \sigma^{s}_{\cdot i}\right) = \ddot{\sigma}^{lj} + \beta^{lj} \ddot{\Theta}, \end{gathered} \tag{1.6}$$

where $H_1 = \alpha_{ij} \beta^{ij}$ is an invariant.

In the absence of temperature effects, the second set of equations (1.6) coincides with the equations obtained in [2].

We will examine a semi-infinite thermoelastic medium at rest, bounded by the surface S. Initially ($t = 0$), the surface S is subjected to temperature perturbations, the displacement velocities are given on part of the surface S_u, and on the rest of the surface S_σ the stresses are known:

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$$
\begin{gathered}
\sigma^{ij}\nu_j|_{S_\sigma} = N^i(x^\alpha)\,Q(t)\,H(t),\\
v^i|_{S_u} = V^i(x^\alpha)\,Q(t)\,H(t), \quad \Theta|_S = F(x^\alpha)\,Q(t)\,H(t),\\
\sigma^{ij},\ u^i,\ \Theta \to 0 \quad \text{for} \quad |x| \to \infty,\\
\sigma^{ij} = \dot\sigma^{ij} = u^i = \dot u^i = \Theta = \dot\Theta = 0 \quad \text{for} \quad t = 0,
\end{gathered} \tag{1.7}
$$

where $v^i = \dot u^i$ is the velocity of displacement; $|x|$ is the distance from the surface S.

We will assume that the function Q is an arbitrary generalized function of time for which a Laplace transformation exists. The transform representation of Q has the form [1]

$$
\tilde Q = \sum_{n=n_*}^{\infty} \frac{Q_{(n)}}{p^{n+1}}. \tag{1.8}
$$

In order to determine the solution of the boundary value problem (1.6) and (1.7), we will use the representation of the solution from [1, 2]

$$
\left\{
\begin{aligned}
\sigma^{ij} &= z^{ij}(x^\alpha, \Omega)\,H(\Omega) = \sum_{n=n_0}^{-1} z^{ij}_{(n)}(x^\alpha)\,\delta^{-(1+n)}(\Omega) + \sum_{n=0}^{\infty} \frac{z^{ij}_{(n)}(x^\alpha)\,\Omega^n}{n!} H(\Omega),\\
u^j &= f^j(x^\alpha, \Omega)\,H(\Omega) = \sum_{n=n_1+1}^{-1} f^j_{(n)}(x^\alpha)\,\delta^{-(1+n)}(\Omega) + \sum_{n=0}^{\infty} \frac{f^j_{(n)}(x^\alpha)\,\Omega^n}{n!} H(\Omega),\\
\Theta &= T(x^\alpha, \Omega)\,H(\Omega) = \sum_{n=n_2}^{-1} T_{(n)}(x^\alpha)\,\delta^{-(1+n)}(\Omega) + \sum_{n=0}^{\infty} \frac{T_{(n)}(x^\alpha)\,\Omega^n}{n!} H(\Omega);
\end{aligned}
\right. \tag{1.9}
$$

$$
\Omega = t - \omega(x^\alpha) \tag{1.10}
$$

or in terms of transforms

$$
\begin{gathered}
\tilde\sigma^{ij} = \sum_{n=n_0}^{\infty} \frac{z^{ij}_{(n)}}{p^{n+1}} e^{-p\omega}, \quad \tilde u^j = \sum_{n=n_1+1}^{\infty} \frac{f^j_{(n)}}{p^{n+1}} e^{-p\omega},\\
\tilde\Theta = \sum_{n=n_2}^{\infty} \frac{T_{(n)}}{p^{n+1}} e^{-p\omega}.
\end{gathered} \tag{1.11}
$$

The quantities n_0, n_1, and n_2 can have negative, but finite, values. The function $\omega(x^\alpha)$ is a smooth function of coordinates that is not known beforehand; p is the Laplace transform variable.

2. Let us determine the conditions for dynamic compatibility of discontinuities for the system of equations of thermoelasticity. Substituting (1.11) into the transformed equations (1.3), taking into account the initial values, collecting terms with identical powers of p, we obtain

$$
\rho f^j_{(n)} = \nabla_i z^{ij}_{(n-2)} - z^{ij}_{(n-1)}\omega_{,i}, \tag{2.1}
$$

and substituting (1.11) into the transformed equations (1.5), we have

$$
z^{ij}_{(n)} = C^{ijkl}(\nabla_k f_{l(n)} - \omega_{,k} f_{l(n+1)}) - \beta^{ij} T_{(n)}. \tag{2.2}
$$

We will refer to the minimum order of the discontinuity of the function represented by a series of the type (1.9) as that value of n for which the nonzero terms of the expansion begin.

It follows from (2.1) that $n_1 \leqslant n_0$, and from (2.2), then, $n_2 \geqslant n_0$. If $T_{(n_0)} \equiv 0$, then $n_1 = n_0$. In particular, in the absence of thermal effects the minimum order of the discontinuity in the displacement is always one unit greater than the order of the discontinuity of the stress. If the orders of the minimum discontinuities in stress and temperature coincide, then exceptional situations can occur.

Let us consider (2.1) for $n = n_0 + 1$:

$$\rho f^j_{(n_0+1)} = -\omega_{,i} z^{ij}_{(n_0)}. \tag{2.3}$$

Let $f^j_{(n_0+1)} = 0$. In accordance with (2.3), this is equivalent to the fact that the minimum discontinuity of the stress vector [2] on the wave front equals zero. In this case, from (2.2)

$$z^{ij}_{(n_0)} = -\beta^{ij} T_{(n_0)}. \tag{2.4}$$

Therefore, if the surface of the wave front is free of stress (the values $n_0 = 0$), then the discontinuity of the displacement velocity also is absent ($n_1 \geqslant n_0 + 1$), while the components of the discontinuity of the stress tensor on surfaces not coinciding with the surface of the front are determined by Eqs. (2.4).

Substituting (1.11) into the transformed equations (1.6), taking into account the zero initial data, and collecting terms with identical powers of p, we obtain recurrence equations of the form

$$\begin{gathered} B_1(T_{(n-2)}) - B_2\left(T_{(n-1)}, z^{ij}_{(n-1)}\right) = B_3\left(T_{(n)}, z^{ij}_{(n)}\right), \\ B_4^{kl}\left(z^{ij}_{(n-2)}\right) + B_5^{kl}\left(z^{ij}_{(n-1)}\right) = B_6^{kl}\left(T_{(n)}, z^{ij}_{(n)}\right), \\ n = n_0, \quad n_0 + 1, \ldots, \end{gathered} \tag{2.5}$$

where the corresponding operators are defined by the equations

$$\begin{gathered} B_3(T, z^{ij}) = [\tau(c_\varepsilon + \eta T_0 H_1) - k^{ij}\omega_{,i}\omega_{,j}]T + \eta T_0 \tau \alpha_{kl} z^{kl}, \\ B_6^{kl}\left(T, z^{ij}\right) = \rho^{-1} C^{klij}\omega_{,i}\omega_{,s} z^{s}_{\cdot j} - z^{kl} - \beta^{kl} T, \\ B_2(T, z^{ij}) = \nabla_j(k^{ij}\omega_{,i} T) + k^{ij}\omega_{,j} T_{,i} + (c_\varepsilon + \eta T_0 H_1)T + \eta T_0 \alpha_{kl} z^{kl}, \\ B_5^{kl}\left(z^{ij}\right) = C^{klij}\{\nabla_i(\rho^{-1}\omega_{,s} z^{s}_{\cdot j}) + \rho^{-1}\omega_{,i}\nabla_s z^{s}_{\cdot j}\}, \\ B_1(T) = \nabla_j\left(k^{ij} T_{,i}\right), \quad B_4^{kl}\left(z^{ij}\right) = -C^{klij}\nabla_i\left(\rho^{-1}\nabla_s z^{s}_{\cdot j}\right). \end{gathered}$$

3. Let us examine Eqs. (2.5) for $n = n_0$. By the method of corresponding [2] linear transformations of the variables $z^{ij}_{(n_0)}$, as well as the rows, the matrix with unknown data for homogeneous algebraic equations can be put into the form

$$\begin{array}{ccccc} \zeta^l_{(n_0)} & T_{(n_0)} & z^{12}_{(n_0)} & z^{13}_{(n_0)} & z^{23}_{(n_0)} \end{array}$$

$$\left(\begin{array}{c|c|ccc} \rho^{-1} C^{ijk\cdot}_{\cdots l}\omega_{,j}\omega_{,k} - \delta^i_l & -\beta^{ik}\omega_{,k} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ \hline \rho^{-1}\eta\tau T_0 \beta^{k\cdot}_{\cdot l}\omega_{,k} & \tau c_\varepsilon - k^{sj}\omega_{,s}\omega_{,j} & 0 & 0 & 0 \\ \hline \rho^{-1} C^{12k\cdot}_{\cdots l}\omega_{,k} & -\beta^{12} & -1 & 0 & 0 \\ \rho^{-1} C^{13k\cdot}_{\cdots l}\omega_{,k} & -\beta^{13} & 0 & -1 & 0 \\ \rho^{-1} C^{23k\cdot}_{\cdots l}\omega_{,k} & -\beta^{23} & 0 & 0 & -1 \end{array}\right). \tag{3.1}$$

Here,

$$\zeta^l_{(n_0)} = z^{li}_{(n_0)}\omega_{,i} = R^l_{(n_0)} G_n^{-1}, \tag{3.2}$$

$R^l_{(n)}$ are the coefficients of the expansion of the stress vector on the wave front in a series similar to the series (1.9); G_n is the velocity of propagation of the wave front along the normal to the front. In (3.1), the variables that stand before the columns are written above the columns of the matrix.

The transformation (3.2) can be inverted (we omit the index n_0):

$$\begin{gathered} z^{11} = \omega_{,1}^{-1}\left(\zeta^1 - \omega_{,2} z^{12} - \omega_{,3} z^{13}\right), \\ z^{22} = \omega_{,2}^{-1}\left(\zeta^2 - \omega_{,1} z^{12} - \omega_{,3} z^{23}\right), \quad z^{33} = \omega_{,3}^{-1}\left(\zeta^3 - \omega_{,1} z^{13} - \omega_{,2} z^{23}\right). \end{gathered}$$

Carrying out similar linear transformations for Eqs. (2.5), for values $n = n_0 + 1, \ldots$ we transform the recurrence equations into the form

$$R_1^k(T_{(n)}, \zeta^i_{(n)}) = R_2^k(T_{(n-1)}, \zeta^i_{(n-1)}) + R_3^k(T_{(n-2)}, \zeta^i_{(n-2)}),$$
$$k = 1, 2, \ldots, 7,$$
$$\text{where} \quad R_m^k(T, \zeta^i) = F_m^{kl}(T, \zeta^i)\,\omega_{,l} \quad \text{for} \quad k = 1, 2, 3;$$
$$R_1^4(T, \zeta^i) = \rho^{-1}\eta\tau T_0 \beta^{k\cdot}_{\cdot l}\omega_{,k}\zeta^l - (k^{sj}\omega_{,s}\omega_{,j} - \tau c_\varepsilon)\,T;$$
$$R_2^4(T, \zeta^i) = \beta^{i\cdot}_{\cdot j}\eta\tau T_0\rho^{-1}\{\nabla_i(\rho^{-1}\zeta^j) + \rho^{-1}\omega_{,i}\nabla_s z^{sj}\} - \nabla_j(k^{ij}\omega_{,i}T) -$$
$$- k^{ij}\omega_{,j}T_{,i} - (c_\varepsilon + \eta T_0 H_1)\,T - \eta T_0 \alpha_{kl} z^{kl}; \tag{3.3}$$
$$R_3^4(T, \zeta^i) = \nabla_j(k^{ij}T_{,i}) - \beta^{i\cdot}_{\cdot j}\eta\tau T_0\rho^{-1}\nabla_i(\rho^{-1}\nabla_s z^{sj});$$
$$R_j^m(T, \zeta^i) = F_j^{kl}(T, \zeta^i) \quad (m = 5 \text{ when } kl = 12; \quad m = 6,$$
$$\text{when } kl = 13; \quad m = 7 \text{ when } kl = 23);$$
$$F_1^{kl}(T, \zeta^i) = \rho^{-1}C^{kls\cdot}_{\cdots j}\omega_{,s}\zeta^j - \beta^{kl}T - \delta^k_j\delta^l_r z^{jr},$$
$$F_2^{kl}(T, \zeta^i) = C^{kli\cdot}_{\cdots j}\{\nabla_i(\rho^{-1}\zeta^j) + \rho^{-1}\omega_{,i}\nabla_s z^{sj}\},$$
$$F_3^{kl}(T, \zeta^i) = -C^{kli\cdot}_{\cdots j}\nabla_i(\rho^{-1}\nabla_s z^{sj}).$$

In addition, $z^{ij}_{(n)}$ are expressed recurrently in terms of $\zeta^i_{(n)}, \zeta^i_{(n-1)}, \ldots$ according to the equation

$$z^{ij}_{(n)} = \rho^{-1}C^{ijk\cdot}_{\cdots l}\omega_{,k}\zeta^l_{(n)} - \beta^{ij}T_{(n)} - F_2^{ij}(T_{(n-1)}, \zeta^l_{(n-1)}) - F_3^{ij}(T_{(n-2)}, \zeta^l_{(n-2)}). \tag{3.4}$$

It is immediately evident from Eq. (3.4) that six components of the stress tensor (z^{ij}) are determined according to the three known components of the stress vector (ζ^j) and the temperature only by means of algebraic operations and differentiation. Using similar procedures, the displacement vector is determined by relations (2.1). In this connection, we reduce the solution of the boundary value problem (1.6), (1.7) to the problem of determining $\zeta^j_{(n)}$, $T_{(n)}$ ($n = n_0, \ldots$). The equations of motion in these variables have already been obtained; this is Eq. (3.3) for the values k = 1, 2, 3, 4, in which $z^{ij}_{(n)}$ are expressed in terms of Eqs. (3.4). We note that in these equations for n = h the right sides in general will depend on the expansion coefficients with numbers h − 1, h − 2, ..., n_0.

Let us obtain the boundary conditions. Since all wave fronts are formed on the surface S, using the equations [2]

$$\nu_j = -\omega_{,j}G_n, \quad G_n = -|\nabla\omega|^{-1},$$

we rewrite the boundary conditions (1.7) in the form

$$\sum_{j=1}^{L} z^{ij}_{(n)}\omega^{(\gamma)}_{,j}G^{(\gamma)}_n\big|_{S_\sigma} = N^i Q_{(n)},$$
$$\sum_{\gamma=1}^{L} \rho^{-1}\left[\nabla_j z^{ij(\gamma)}_{(n-1)} - \zeta^{i(\gamma)}_{(n)}\right]\big|_{S_u} = V^i Q_{(n)}, \quad \sum_{\gamma=1}^{L} T^{(\gamma)}_{(n)}\big|_S = F Q_{(n)}, \tag{3.5}$$

where γ corresponds to the number of the front; L is the number of wave fronts in the problem.

4. For further investigations of problems of solubility of the recurrence equations, we will probe the hyperbolic nature of the system of equations of thermoelasticity (1.1)-(1.5). We will carry out the analysis in a Cartesian system of coordinates. It can be shown that the matrix of the characteristic determinant of the system of equations of thermoelasticity coincides with the matrix (3.1) (if the characteristic surface is sought in the form $\Omega = 0$). As is well known [5], the condition for the hyperbolic property is equivalent to the condition that at any chosen point in space along any chosen direction there exist exactly 4 possible velocities of propagation of the wave (taking into account the multiplicity of the velocities). Sometimes, in the presence of multiple characteristics, the systems of equations are called weakly hyperbolic [6].

The determinant of the matrix (3.1) coincides to within a factor ρ^3 with the determinant of the 4 × 4 corner minor.

Let us write the auxiliary system of algebraic equations relative to the unknowns p_i (i = 1,..., 4)

$$\sum_{j=1}^{4} \gamma_{ij}p_i = \delta_i p_i, \tag{4.1}$$

where

$$\gamma_{ij} = \gamma_{ji} = a_{ij} = C^{ikjl}\nu_k\nu_l; \quad \gamma_{4i} = \gamma_{i4} = \pi_i = \beta_i^k\nu_k \quad (i, j = 1, 2, 3);$$
$$\gamma_{44} = -\omega_1 = -c_\varepsilon T_0^{-1}; \quad \delta_i = \rho G_n^2; \quad \delta_4 = -\omega_2 G_n^{-2} = -(\tau T_0)^{-1} k^{sj}\nu_j\nu_s G_n^{-2}.$$

The determinant of Eqs. (4.1)

$$I = \begin{vmatrix} a_{ij} - \rho G_n^2 \delta_{ij} & \pi_j \\ \pi_i & \omega_2 G_n^{-2} - \omega_1 \end{vmatrix} \tag{4.2}$$

coincides to within a nonzero factor with the characteristic determinants of the equations of thermoelasticity. We will show that all roots $\mu = G_n^2$ of the equation $I = 0$ are real.

The condition $I = 0$ is equivalent to the existence of nonzero solutions p_j ($j = 1,\ldots, 4$). Let $\mu = \mu_1 + i\mu_2$ be the complex value of one of the roots and the solution p_j corresponding to it, which, generally speaking, is also complex. We will show that $\mu_2 = 0$.

Let us transform Eqs. (4.1) as follows:

$$\sum_{i=1}^{4}\sum_{j=1}^{4} \gamma_{ij} p_j \bar{p}_i = \sum_{i=1}^{4} \delta_i |p_i|^2 = \rho G_n^2 \sum_{i=1}^{3} |p_i|^2 - \omega_2 G_n^{-2} |q_4|^2. \tag{4.3}$$

Here, the bar indicates complex conjugation; $|\ldots|$ indicates the modulus of the complex number.

Since the matrix γ_{ij} ($i, j = 1,\ldots, 4$) is symmetric, the left side of (4.3) is real. For this reason, we equate the imaginary part of the right side of (4.3) to zero:

$$\left[\rho \sum_{i=1}^{3} |p_i|^2 + \omega_2 |q_4|^2 (\mu_1^2 + \mu_2^2)^{-1}\right] \mu_2 = 0.$$

The invariant $\omega_2 = (\tau T_0)^{-1} k^{sj}\nu_j\nu_s$ is a positive definite quadratic form [3]. It follows from here that $\mu_2 \equiv 0$, so that the quantity G_n^2 is real. We will show that the quantity G_n is also real (i.e., $G_n^2 > 0$). For this purpose, let us multiply the fourth row and the fourth column of the matrix of the determinant (4.2) by G_n and write the system of equations of type (4.1) corresponding to its matrix, for which

$$\gamma_{ij} = \gamma_{ji} = a_{ij}, \quad \gamma_{4i} = \gamma_{i4} = \pi_i G_n, \ \gamma_{44} = \omega_2, \quad \delta_i = \rho G_n^2,$$
$$\delta_4 = \omega_1 G_n^2 \quad (i, j = 1, 2, 3).$$

We repeat the calculations similar to (4.3):

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} p_j \bar{p}_i + G_n \sum_{i=1}^{3} \pi_i p_i \bar{p}_4 + G_n \sum_{i=1}^{3} \pi_i p_4 \bar{p}_i + \omega_2 |p_4|^2 =$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} p_j \bar{p}_i + \omega_2 |p_4|^2 + 2G_n \sum_{i=1}^{3} \pi_i \operatorname{Re}(p_i p_4) = \rho G_n^2 \sum_{i=1}^{3} |p_i|^2 + \omega_1 G_n^2 |p_4|^2. \tag{4.4}$$

The right side of (4.4) is real. The sum on the left side is also real. From here we have the fact that G_n is real.

Thus, we have proved that the system of equations of thermoelasticity is hyperbolic. We note that if in order to prove that the system of equations of theory of elasticity ($\Theta \equiv 0$) is hyperbolic it is enough that only the first law of thermodynamics is satisfied, providing symmetry of the coefficients a_{ij}, then for thermoelasticity, in addition, it is necessary that the second law of thermodynamics be satisfied in order that the quadratic form ω_2 be positive definite.

Since the system of equations of thermoelasticity has been shown to be hyperbolic, practically repeating word for word the discussion from [1, 2], it can be shown that for each value of n the solution of the recurrence equations (2.5) reduces to a system consisting of four differential equations with first order partial derivatives. For values $n \geqslant n_0 + 1$, the equations will have in general an inhomogeneous right side. The initial conditions for these differential equations are determined from (3.5).

Thus, the problem of determining the expansion coefficients for the series (1.9) has been reduced to Cauchy's problems for a system of differential equations with first order partial derivatives. It is assumed that all necessary conditions providing for the existence and uniqueness of the solutions of these problems are satisfied. The uniqueness in the class of wave solutions of the problem (1.6), (1.7) follows from here, and when the series (1.9) converge, the existence of the solutions follows as well.

We note that the system of equations (1.6) reduces in a recurrent manner to the solution of Cauchy's problems with the most general right sides of conditions (1.7). The only limitation is that the solution belong to the space of Laplace transformations.

5. Let us study the special case of the solvability of the recurrence equations (2.5) for the uncoupled problem $(\eta = 0, \tau \neq 0)$, when the characteristic determinant has the form

$$\left(\tau c_\varepsilon - k^{ij}\omega_{,i}\omega_{,j}\right) \det \left| C^{ijk\cdot}_{\ldots l}\omega_{,j}\omega_{,k} - \rho\delta^i_l \right| = 0. \tag{5.1}$$

If ω is found from the condition that the first cofactor equals zero, then the front of the temperature wave is determined. For such a value of ω, the temperature is determined independently of the stress. If for given ω the second cofactor in (5.1) differs from zero, i.e., the front of the temperature wave does not coincide with any front of the elastic wave, then in order to determine the jumps of the stress vector for each value of n there exists a nondegenerate system of linear algebraic equations with the right side known from the solution to the temperature problem. Thus, on the temperature front, the stresses are determined from the solution of these algebraic equations. The rest of the solutions ω, determined from the fact that the second cofactor of Eq. (5.1) equals zero, will give the fronts of the elastic waves. Since the first cofactor in (5.1) is not equal to zero, from (2.5) $(\eta = 0)$ it follows that on these fronts $\Theta = 0$. Therefore, the discontinuities of stresses on these fronts are found by the same method as in [2]. Thus, the stress and temperature are discontinuous on the front of the temperature wave when only the stresses are discontinuous on the front of the elastic wave. In the case that the temperature front coincides with one of the elastic wave fronts, on this front the equation

$$\left(C^{ijk\cdot}_{\ldots l}\omega_{,k}\omega_{,j} - \delta^i_l\right)\zeta^l_{(n_0)} = \beta^{ij}\omega_{,j}T_{(n_0)} \tag{5.2}$$

with a degenerate matrix with unknowns $\zeta^l_{(n_0)}$ must have a nontrivial solution. For the one-dimensional problem $\left(\zeta^1_{(n_0)} \neq 0,\ \ \zeta^2_{(n_0)} = \zeta^3_{(n_0)} = 0\right)$, it is necessary that $T_{(n_0)} = 0$, i.e., the minimum order of the discontinuity in temperature on the given front must be one unit less than the order of the stress discontinuity.

For a multidimensional problem, in general, it is also necessary that $T_{(n_0)} = 0$, but in specific situations this is not necessary. It is sufficient that the solution for the temperature and anisotropy of the medium be such that in (5.2) the rank of the matrix in front of $\zeta^l_{(n_0)}$ and the rank of the enlarged matrix coincide. It is possible to construct simple examples.

We note especially that the limitations between values of n_0 and n_2 in the expansions (1.9) occur only for solutions (actually for particular solutions) of systems of recurrent differential equations (2.5), but not for the boundary conditions (1.4). The latter can be given independently.

6. Let us examine the condition for dynamic compactness of the coupled thermoelastic waves, when the boundary loads on the entire surface S are given only for stresses and temperature, i.e., in (1.7) $S_u = \varnothing$ is an empty set.

<u>THEOREM.</u> The condition for dynamic compactness of waves for the system of equations of thermoelasticity (1.6) with the loads (1.7) (for $S_u = \varnothing$) consists of the fact that the following the system of equalities is satisfied (on all fronts simultaneously):

$$\begin{gathered}\left(\rho^{-1}C^{kls\cdot}_{\ldots r}\omega_{,s}\omega_{,k} - \delta^l_r\right)\zeta^r - \beta^{ik}\omega_{,k}T = 0,\\ F^{kl}_2\left(T, \zeta^i\right) = F^{kl}_3\left(T, \zeta^i\right) = R^4_m\left(T, \zeta^i\right) = 0 \qquad (m = 1, 2, 3).\end{gathered} \tag{6.1}$$

In addition, in relations (6.1),

$$z^{ij} = \rho^{-1}C^{ijk\cdot}_{\ldots l}\omega_{,k}\zeta^l - \beta^{ij}T. \tag{6.2}$$

Proof. Let us use the lemma in [1, 2], stating that the condition for dynamic compactness is equivalent to the fact that an impulsive (δ-function of time) boundary load on the surface S corresponds to a solution as a sum of waves each of which at a fixed time represents a δ-function of coordinates.

Let only a single term of the series $Q_{(-1)}$ in the expansions (1.8) differ from zero. We will prove that when the conditions (6.1) are satisfied, the series (1.9) for the stresses and temperature will also contain a single term.

First, we note that Eq. (6.2) is a result of relations (3.4), (6.1), asserting that if the series of type (1.9) for the stress vector consists only of a single coefficient, then the similar series for the stress tensor also contains only a single term. For this reason, in order to prove the lemma (and therefore, the theorem as well) it is sufficient to show that when conditions (6.1) are satisfied one of the terms of the expansion in the series (1.8) of the function Q corresponds to a series consisting of a single term for the stress vector and temperature.

Let us begin the solution of the recurrent system of equations (3.3). After determining $\zeta^l_{(-1)}$ and $T_{(-1)}$, in order to determine $\zeta^l_{(0)}, T_{(0)}, \zeta^l_{(1)}$, $T_{(1)}$, and so on at each stage of the calculation we will have a Cauchy problem for a system of linear first order differential equations, whose only solution is the null solution. Thus, the lemma is true, and therefore, the theorem is proved.

Remark. The theorem is true only in the case that for the expansions (1.9) $n_0 = n_2$. In the case that $n_0 > n_2$, conditions (6.1) will take on a somewhat different form. In the present work, in view of the limited space, this problem will not be considered.

7. Let us examine somewhat transformed conditions of dynamic compactness (6.1) for an orthotropic inhomogeneous half-space, when uncoupled stress σ^{11} and temperature Θ waves propagate along the direction of one of the axes of symmetry of the material x_1:

$$\left(\rho^{-1}a_{11}\omega_{,1}^2 - 1\right) z^{11} - \beta^{11}T = 0; \tag{7.1}$$

$$a_{11}(\rho^{-1}z^{11}\omega_{,1})_{,1} + \rho^{-1}\omega_{,1}z^{11} = 0; \tag{7.2}$$

$$\left(\rho^{-1}z^{11}_{,1}\right)_{,1} = 0; \tag{7.3}$$

$$\left(\tau c_\varepsilon - k_{11}\omega_{,1}^2\right) T = 0; \tag{7.4}$$

$$c_\varepsilon T + (k_{11}\omega_{,1}T)_{,1} + k_{11}\omega_{,1}T_{,1} = 0; \tag{7.5}$$

$$(k_{11}T_{,1})_{,1} = 0, \tag{7.6}$$

where $a_{11} = C_{1111}$; z^{11} and T are the discontinuities in stress and temperature.

From (7.1) and (7.4), we obtain two solutions for ω_1^2:

$$\omega_{,1}^2 = \tau c_\varepsilon k_{11}^{-1},\ T = \varphi z^{11},\quad \varphi = \frac{\rho^{-1}\omega_{,1}^2 - 1}{\beta_{11}}; \tag{7.7}$$

$$\omega_{,1}^2 = \rho a_{11}^{-1},\ T = 0. \tag{7.8}$$

When the relations (7.7) are realized (the front of the temperature wave), we have from (7.5)

$$T = C_1 (k_{11}\omega_{,1})^{-1/2} \exp\left\{- {}^1/_2 \int c_\varepsilon (k_{11}\omega_{,1})^{-1} dx_1\right\}. \tag{7.9}$$

Here and in what follows, the symbol C_α denotes the arbitrary constants. From (7.2), we have

$$z^{11} = C_2 \left(\rho\omega_{,1}^{-1}\right)^{1/2}. \tag{7.10}$$

From (7.3) and (7.6), it follows that

$$T = C_3 \int k_{11}^{-1} dx_1 + C_4,\quad z^{11} = C_5 \int \rho dx_1 + C_6. \tag{7.11}$$

From the requirement that the conditions (7.7), (7.9)-(7.11) coincide, we obtain

$$C_1 (k_{11}\omega_{,1})^{-1/2} \exp\left\{- {}^1/_2 \int c_\varepsilon (k_{11}\omega_{,1})^{-1} dx_1\right\} = C_3 \int k_{11}^{-1} dx_1 + C_4; \tag{7.12}$$

$$C_2(\rho\omega_{,1}^{-1})^{1/2} = C_5\int \rho dx_1 + C_6; \tag{7.13}$$

$$C_3\int k_{11}^{-1}dx_1 + C_4 = \varphi C_2(\rho\omega_{,1}^{-1})^{1/2}. \tag{7.14}$$

With the realization (7.8) (the case of an elastic wave), the compatibility of the relations (7.1)-(7.6) requires that only a single equality be satisfied:

$$C_2(\rho a_{11})^{1/4} = C_5\int \rho dx_1 + C_6. \tag{7.15}$$

Relations (7.12)-(7.15) impose limitations on the structure of the medium and, because of the constants C_α, on the type of solution as well. For dynamic compactness, only a single temperature wave is sufficient for satisfying Eq. (7.12), while for dynamic compactness of only the elastic wave, the equality (7.15). We note that (7.13) cannot be satisfied for properties of the half-space that do not depend on the coordinate x_1. For this reason, the temperature wave in the uniform half-space propagates with diffusion. On the other hand, the elastic one-dimensional wave propagates in the half-space without diffusion.

8. Let us examine the problem of controlling the number of waves. Let $n = n_0$ in Eqs. (3.3). We will assume that all propagation velocities of the wave fronts G_n are aliquant, then, for each solution $\omega_{(\gamma)}$ the rank of the corner minor of the matrix (3.1) equals 3. Therefore, the system of linear algebraic equations with an inhomogeneous right side

$$\left(\rho^{-1}G_{n(\gamma)}^{-2}C^{ijk}_{\cdot\cdot\cdot l}\nu_{j(\gamma)}\nu_{k(\gamma)} - \delta^i_l\right)\zeta^l_{(\gamma)} = \beta^{ik}\nu_{k(\gamma)}G_{n(\gamma)}^{-1}T_{(\gamma)}$$

can be solved for $\zeta^l_{(\gamma)}$:

$$\zeta^l_{(\gamma)} = \eta^l_\gamma T_{(\gamma)} \tag{8.1}$$

(in the relations used, the index n_0 is omitted; γ varies from 1 to 4).

The sums of the particular solutions must satisfy the boundary conditions (in the forces and temperature):

$$\sum_{\gamma=1}^{4}\zeta^l_{(\gamma)}G_{n(\gamma)}\Big|_S = N^l_0, \quad \sum_{\gamma=1}^{4}T_{(\gamma)}\Big|_S = F_0. \tag{8.2}$$

Here, $N^l_0 = N^l Q_{(n_0)}$ and $F_0 = FQ_{(n_0)}$. Substituting (8.1) into (8.2), we obtain

$$\sum_{\gamma=1}^{4}\eta^l_\gamma G_{n(\gamma)}T_{(\gamma)}\Big|_S = N^l_0, \quad \sum_{\gamma=1}^{4}T_{(\gamma)}\Big|_S = F_0. \tag{8.3}$$

In view of the uniqueness of the solution of the corresponding Cauchy problems, the system of equations (8.3) can be solved for $T_{(\gamma)}|_S$ in the form

$$T_{(\gamma)}|_S = \pi_{\gamma l}N^l_0 + \pi_{\gamma 4}F_0,$$

where the coefficients $\pi_{\gamma l}$ and $\pi_{\gamma 4}$ are simply expressed in terms of $\eta^l_\gamma G_{n(\gamma)}$.

Due to the aliquant nature of the wave fronts, only a single independent function $T_{(\gamma)}$ exists on each solution. In order to determine $T_{(n_0)}$ on each front, we have a Cauchy problem for a system of homogeneous first order differential equations. In view of the uniqueness of the solution of Cauchy's problem, we have the result: If the following condition is satisfied on the surface S for a front with number

$$\pi_{\gamma l}N^l_0 + \pi_{\gamma 4}F_0 = 0, \tag{8.4}$$

then on this front the minimum order of the discontinuity of the stress vector and temperature is one unit greater than the minimum order of the discontinuity for the boundary conditions. If, in addition, on these fronts, conditions of dynamic compactness (6.1) are satisfied, then fronts with this number will not exist. In the latter case, in (8.4), N_0^l and F_0 should be understood as N^i and F.

We note that relation (8.4) cannot be satisfied for all four fronts.

Let relation (8.4) exist for $\gamma = 1, 2, 3$, i.e., $T_{(1)} = T_{(2)} = T_{(3)} \equiv 0$. Then, relation (8.3) will have the form

$$\eta_4^l G_{n(4)} T_{(4)} \big|_S = N_0^l.$$

Let us determine the expression $\eta_4^l G_{n(4)}$. We have

$$-\zeta_{(4)}^i = \beta^{ik} \nu_{k(4)} G_{n(4)}^{-1} T_{(4)} - G_{n(4)}^{-2} C^{ijk}_{\cdots l} \nu_{j(4)} \nu_{k(4)} \zeta_{(4)}^l. \tag{8.5}$$

Substituting into (8.5) the expression from (8.2), having the form

$$\zeta_{(4)}^l \big|_S = N_0^l G_{n(4)}^{-1} \big|_S, \quad T_{(4)} \big|_S = F_0,$$

we obtain

$$N_0^l = \left\{ \beta^{lk} \nu_{k(4)} T_{(4)} - \rho^{-1} G_{n(4)}^{-2} C^{ljk\cdot}_{\cdots s} \nu_{j(4)} \nu_{k(4)} N_0^s \right\} \Big|_S. \tag{8.6}$$

Let us examine the degenerate case. Let the value $\tau \to 0$. From the form of the matrix (3.1), it follows that $\omega_{(4)} \to 0$, and from here the velocity of propagation of the front $G_{n(4)} \to \infty$. Then, from (8.6) we obtain that in this limiting case

$$N_0^l = \beta^{lk} \nu_k F_0. \tag{8.7}$$

COROLLARY. If, with a thermal or mechanical impact on the surface S, the boundary loads for the coupled or uncoupled problem of thermoelasticity (without taking into account the finite velocity of propagation of the heat flux) satisfy conditions (8.7), then the stress vector is not discontinuous on the wave front.

The validity of the latter corollary can be verified for particular examples of a heat impulse on the surface of an isotropic half-space [3] and a sphere [7]. Separate problems on controlling thermoelastic waves were solved previously in [8].

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